

On the Fix-Points of Composite Transcendental Entire Functions

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I. INTRODUCTION

A transcendental entire function $F(z)$ is called composite iff $F(z) \equiv f(g(z))$ for some nonlinear entire functions f and g . A point z_0 in the complex plane is called a fix-point of F iff $F(z_0) = z_0$. Gross [3] conjectured that every composite transcendental entire function must have infinitely many fix-points. It has been shown, thus far, that the conjecture is valid for finite order entire functions by several authors (e.g., [2, 4, 7]) via different approaches. In a previous paper [11], we investigated the fix-points of a certain type of infinite order composite entire function. There, it is required that g be an entire function of positive finite order with a finite Nevanlinna exceptional value and f be an arbitrary entire function such that the hyperorder of $f(g)$ is less than the order of g . In this note we shall treat some other class of composite entire functions. The tools are, as usual, based on Nevanlinna's theory of meromorphic functions and an elegant result due to Steinmetz [8, Corollary 1]. Our result is the following theorem.

THEOREM. *Let $F(z) \equiv f(g(z))$ be an infinite order composite entire function, where the order ρ_f of f is either less than $1/2$ or an irrational number and g satisfies, for some constant $c > 1$,*

$$\overline{\lim}_{r \rightarrow \infty} T(cr, g)/T(r, g) \leq k(c) < \infty.$$

Then F and $g(f)$ both must have infinitely many fix-points.

II. PRELIMINARY LEMMAS

In this section, we shall quote and prove some lemmas that will be employed in the proof of the theorem. It is assumed that the reader is familiar with the standard notations used in Nevanlinna theory.

LEMMA 1 (Clunie [1]). *If both f and g are entire functions with $g(0) = 0$, then for $r \geq 0$*

$$\log M(r, f(g)) \geq \log M(c(\rho) M(\rho r, g), f),$$

where $0 < \rho < 1$ and $c(\rho) = (1 - \rho)^2/4$.

LEMMA 2 [5, p. 8]. *Let f be an entire function. Then for $0 \leq r < R$*

$$T(r, f) \leq \log^+ M(r, f) \leq \frac{R+r}{R-r} T(R, f).$$

This and the fact: $\log M(r, f(g)) \leq \log M(M(r, g), f)$ leads to:

LEMMA 3 [6, p. 374]. $T(r, f(g)) \leq 3T(2M(r, g), f)$.

Using the above lemmas we can derive the following:

LEMMA 4. *Let $f(z)$ be an entire function of finite order and g be an arbitrary entire function satisfying, for some constant $c > 1$, $\lim T(cr, g)/T(r, g) \leq k(c) < \infty$. If, for some entire function α , $T(r, e^\alpha) \sim T(r, f(g))$ as $r \rightarrow \infty$, then*

$$(i) \quad \overline{\lim}_{r \rightarrow \infty} T(r, \alpha)/T(r, g) \leq d \quad (0 \leq d < \infty)$$

and

$$(ii) \quad \overline{\lim}_{\substack{r \rightarrow \infty \\ r \notin E}} T(r, \alpha')/T(r, g) \leq d,$$

where E denotes a set of r values of finite length.

Proof of Lemma 4. By assumption, for an arbitrary small quantity ε , we have for $r > r_0$

$$\begin{aligned} 1 - \varepsilon &\leq \frac{\log T(r, f(g))}{\log T(r, e^\alpha)} \leq \frac{\log 3T(2M(r, g), f)}{\log T(r, e^\alpha)} \\ &= \{\log 3T(2M(r, g), f)/\log 2M(r, g)\} / \{\log T(r, e^\alpha)/\log 2M(r, g)\}. \end{aligned}$$

It follows that for $r > r_1$

$$1 - \varepsilon \leq \frac{(\rho_f + \varepsilon) \log 2M(r, g)}{\log T(r, e^\alpha)}. \quad (1)$$

Now

$$\begin{aligned}
 \log T(r, e^z) &\geq \log \log \left\{ 1/3 M\left(\frac{r}{2}, e^z\right) \right\} \\
 &= \log \log M\left(\frac{r}{2}, e^z\right) + O(1) \\
 &\geq \log \log M\left(c(\rho) M\left(\frac{\rho r}{2}, \alpha\right), e^z\right) + O(1). \quad (2)
 \end{aligned}$$

Thus (1) and (2) yields, for $r > r_2$,

$$\begin{aligned}
 \frac{1-\varepsilon}{\rho_f + \varepsilon} &\leq \frac{\log 2 + \log M(r, g)}{\log(c\rho M(\rho r/2, \alpha)) + O(1)} \\
 &\leq \frac{\log 2 + \log M(r, g)}{T(\rho r/2, \alpha) + O(1)} \leq \frac{3T(2r, g) + O(1)}{T(\rho r/2, \alpha) + O(1)}. \quad (3)
 \end{aligned}$$

Now using the assumption that $\overline{\lim}_{r \rightarrow \infty} T(cr, g)/T(r, g) \leq k(c) < \infty$, $c > 1$, we derive from the above

$$\frac{1-\varepsilon}{\rho_f + \varepsilon} < \frac{3k(4/\rho) T(\rho r/2, g) + O(1)}{T(\rho r/2, \alpha) + O(1)}. \quad (4)$$

Since ρ can be fixed, it follows from the above

$$\overline{\lim}_{r \rightarrow \infty} \frac{T(r, \alpha)}{T(r, g)} \leq d \quad (0 \leq d < \infty).$$

Assertion (i) is thus proved. Assertion (ii) follows from assertion (i) and the fact that $T(r, \alpha') \leq (1 + O(1)) T(r, \alpha)$ as $r \rightarrow \infty$, $r \notin E$, where E is a set of r values of finite length.

III. PROOF OF THE THEOREM

Suppose that $F(z) \equiv f(g(z))$ has only finitely many fix-points. That is,

$$f(g) - z \equiv pe^\alpha, \quad (5)$$

where p is a polynomial and α an entire function.

We now differentiate (5) and obtain

$$f'(g) g' - 1 \equiv (p' + p\alpha') e^\alpha. \quad (6)$$

Eliminating e^z from (5) and (6) we get

$$\frac{f'(g) g' - 1}{f(g) - z} \equiv \alpha' + \frac{p'}{p}$$

or

$$f'(g) g' - f(g) \left(\frac{p'}{p} + \alpha' \right) + z \left(\frac{p'}{p} + \alpha' \right) - 1 \equiv 0. \quad (7)$$

Now according to assertion (ii) of Lemma 4, all the coefficients $h_0(z) \equiv g'$, $h_1(z) \equiv -(p'/p + \alpha')$, and $h_2(z) \equiv z(p'/p + \alpha') - 1$ satisfy $T(r, h_i) \leq kT(r, g)$ ($i=0, 1, 2$) as $r \rightarrow \infty$, $r \notin E$, where k is a suitable positive constant. Therefore, by applying Steinmetz's result [8, Corollary 1] on identity (7), we conclude that there exist polynomials Q_0 , Q_1 and Q_2 not all identically zero such that

$$Q_0(z) f'(z) + Q_1(z) f(z) + Q_2(z) \equiv 0. \quad (8)$$

Now according to Valiron [10], there showed that for any first-order algebraic differential equation, the order of any entire transcendental solution must be a positive rational number. Also recently Strelitz [9] showed further that the order must be at least $1/2$. It follows from these observations that the order of the entire solution f of Eq. (8) is rational and at least $1/2$, contradicting our hypotheses on ρ_f . Therefore $f(g)$ must have infinitely many fix-points. The assertion about $g(f)$ follows from the fact that $f(g)$ has infinitely many fix-points iff $g(f)$ does.

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